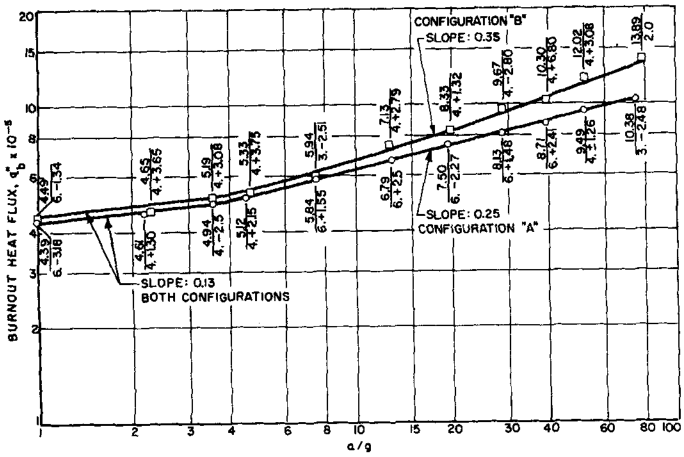

Fig. 3. Pool boiling burnout heat flux (q_b'') vs. a/g for heater configurations A and B (Figure 2). Numbers near data points: average q_b''/number of points averaged; maximum percentage deviation from average.

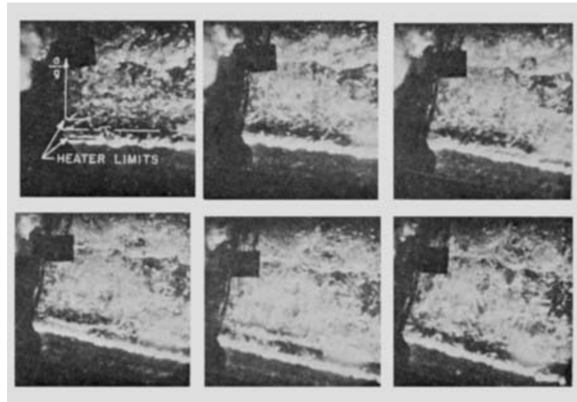

Fig. 4. Sequence of consecutive frames showing wave action of vapor coming from portion of cylindrical heater toward which acceleration directed. Film speed 2,730 frames/sec. q_b'' = 718,000 B.t.u./hr. sq. ft. a/g = 40.1. Note film boiling on top of heater at right. Bottom of heater still in nucleate boiling. Wave moves at average velocity of about 17 ft./sec.

and experimental studies on this interesting phenomenon are continuing.

Figure 4 illustrates a possible reason for the greater resistance to burnout of heater orientation B. Bubbles accumulate at the side of the heater toward which the acceleration is directed. Then they sweep off together, picking up other bubbles as they go. They may sweep the other bubbles away before coalescence can occur. The wave action of the bubbles is more pronounced at high accelerations; this may explain why the difference in burnout heat fluxes between heater orientations A and B is more pronounced at high accelerations.

The effect of orientation may be important in design considerations for swirl flow, where high accelerations are created by the flow itself, and may possibly be helpful in improving understanding of the burnout problem.

ACKNOWLEDGMENT

The authors are grateful to the National Science Foundation for financial help afforded this project.

A Note on Natural Convection Effects in Fully Developed Horizontal Tube Flow

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Effects due to the gravitational field in nonisothermal bounded horizontal flows have been investigated theoretically only quite recently. Morton (*2*) studied fully-developed flow in tubes, and calculations for the problem involving infinite parallel plates are available (*1*). However Morton's results for tubes indicate that natural convection effects vanish as N_{Re} goes to zero in a nonisothermal system. This seems unlikely, and the purpose of the present analysis is to provide a more complete solution which does not depend on the assumption of a constant axial pressure gradient.

A perturbation analysis, which linearizes the momentum equations, shows that additional terms important for slow flows arise owing to the axial density variation. It is interesting to note in advance that minor effects involving the Froude number are also obtained. If the usual assumption that the density is constant except in the body force term is made, the results are the same but the Froude number dependence vanishes. Hence the Froude number effect arises from the inertial density variation. The results of the analysis given here are applicable to fully-developed flows in the sense that the radial and angular velocities are functions of r and θ only, and the axial velocity changes uniformly but very gradually in the direction of flow. For simplicity the viscosity, thermal conductivity, and heat capacity are assumed constant.

The momentum equations in dimensionless cylindrical coordinates, where θ is measured from the center plane of the tube parallel to the earth's surface, are

$$\frac{\rho}{\rho_o}\left[\phi_1\frac{\partial\phi_1}{\partial\epsilon}+\phi_2\frac{\partial\phi_1}{\partial\lambda}+\frac{\phi_3}{\lambda}\frac{\partial\phi_1}{\partial\theta}\right]=-\frac{\partial p}{\partial\epsilon}+\frac{1}{N_{Re}}\nabla^2\phi_1 \quad (1)$$

$$\frac{\rho}{\rho_o}\left[\phi_1\frac{\partial\phi_2}{\partial\epsilon}+\phi_2\frac{\partial\phi_2}{\partial\lambda}+\frac{\phi_3}{\lambda}\frac{\partial\phi_2}{\partial\theta}-\frac{\phi_3^2}{\lambda}\right]=-\frac{\partial p}{\partial\lambda}+\frac{1}{N_{Re}}\left[\nabla^2\phi_2-\frac{\phi^2}{\lambda^2}-\frac{2}{\lambda^2}\frac{\partial\phi_3}{\partial\theta}\right]-\frac{a\,g\sin\theta}{U_o^2}\frac{\rho}{\rho_o} \quad (2)$$

$$\frac{\rho}{\rho_o}\left[\phi_1\frac{\partial\phi_3}{\partial\epsilon}+\phi_2\frac{\partial\phi_3}{\partial\lambda}+\frac{\phi_3}{\lambda}\frac{\partial\phi_3}{\partial\theta}+\frac{\phi_2\phi_3}{\lambda}\right]=-\frac{1}{\lambda}\frac{\partial p}{\partial\theta}+\frac{1}{N_{Re}}\left[\nabla^2\phi_3+\frac{2}{\lambda^2}\frac{\partial\phi_2}{\partial\theta}-\frac{\phi_3}{\lambda^2}\right]-\frac{a\,g\cos\theta}{U_o^2}\frac{\rho}{\rho_o} \quad (3)$$

and the continuity equation is

$$\frac{\partial\left(\frac{\rho}{\rho_o}\lambda\phi_1\right)}{\partial\epsilon}+\frac{\partial\left(\frac{\rho}{\rho_o}\lambda\phi_2\right)}{\partial\lambda}+\frac{\partial\left(\frac{\rho}{\rho_o}\phi_3\right)}{\partial\theta}=0 \quad (4)$$

Also the energy equation is

$$\frac{\rho}{\rho_o}\left[\phi_2\frac{\partial t}{\partial\lambda}+\frac{\phi_3}{\lambda}\frac{\partial t}{\partial\theta}+\phi_1\frac{\partial t}{\partial\epsilon}\right]=\frac{1}{N_{Pe}}\nabla^2 t \quad (5)$$

and the temperature distribution is specified linear in the axial coordinate

$$t-t_o=Aa\epsilon+F(\lambda,\theta) \quad (6)$$

Since the density is assumed to be temperature dependent only, the equation of state is

$$\frac{\rho}{\rho_o} = 1 - \beta\,(t - t_o) \qquad (7)$$

It is convenient to eliminate the pressure gradients from Equation (1) to (3) by differentiating Equation (1) by λ and θ and equating results with the derivatives of Equations (2) and (3) with respect to ϵ. This yields

$$\frac{\partial}{\partial\lambda}\left[\frac{1}{N_{Re}}\nabla^2\phi_1 - \frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\,\phi_1\right] = \frac{1}{N_{Re}}\frac{\partial}{\partial\epsilon}\left[\nabla^2\phi_2 - \frac{\phi_2}{\lambda^2} - \frac{2}{\lambda^2}\frac{\partial\phi_3}{\partial\theta}\right] - \frac{\partial}{\partial\epsilon}\left[\frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\phi_2\right] - \frac{a\,g\sin\theta}{U_o^2}\frac{\partial}{\partial\epsilon}\left(\frac{\rho}{\rho_o}\right) \qquad (8)$$

$$\frac{1}{\lambda}\frac{\partial}{\partial\theta}\left[\frac{1}{N_{Re}}\nabla^2\phi_1 - \frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\phi_1\right] = \frac{1}{N_{Re}}\frac{\partial}{\partial\epsilon}\left[\nabla^2\phi_3 + \frac{2}{\lambda^2}\frac{\partial\phi_2}{\partial\theta} - \frac{\phi_3}{\lambda^2}\right] - \frac{\partial}{\partial\epsilon}\left[\frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\phi_3\right] - \frac{a\,g\cos\theta}{U_o^2}\frac{\partial}{\partial\epsilon}\left(\frac{\rho}{\rho_o}\right) \qquad (9)$$

$$\frac{1}{N_{Re}}\frac{\partial}{\partial\lambda}\left\{\lambda\left[\nabla^2\phi_3 + \frac{2}{\lambda^2}\frac{\partial\phi_2}{\partial\theta} - \frac{\phi_3}{\lambda^2}\right]\right\} - \frac{\partial}{\partial\lambda}\left\{\lambda\left[\frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\phi_3\right]\right\} - \frac{a\,g\cos\theta}{U_o^2}\frac{\partial}{\partial\lambda}\left\{\lambda\frac{\rho}{\rho_o}\right\} = \frac{1}{N_{Re}}\frac{\partial}{\partial\theta}\left[\nabla^2\phi_2 - \frac{\phi_2}{\lambda^2} - \frac{2}{\lambda^2}\frac{\partial\phi_3}{\partial\theta}\right] - \frac{\partial}{\partial\theta}\left[\frac{\rho}{\rho_o}(\vec{V}\cdot\nabla)\phi_2\right] - \frac{a\,g}{U_o^2}\frac{\partial}{\partial\theta}\left(\frac{\rho}{\rho_o}\sin\theta\right) \qquad (10)$$

Pressure gradients have been eliminated in Equations (8) to (10), but to establish a clear physical basis for the results obtained later the form of $\partial p/\partial\epsilon$, with $\phi_1 = \phi_1\,(\theta, \lambda)$ assumed for simplicity, will be deduced directly. When one integrates Equations (2) and (3) with respect to λ and θ respectively and then differentiates with respect to ϵ, the expressions obtained are equivalent if

$$\frac{\partial p}{\partial\epsilon} = h(\epsilon) - N\lambda\sin\theta$$

When one substitutes this expression into Equation (1) and notes that ϕ_1, ϕ_2, and ϕ_3 are not functions of ϵ, it follows that $h(\epsilon)$ = constant.

Clearly Equations (8) to (10) are extremely complicated, but the problem may be made tractable if a perturbation analysis is employed. The expansion parameter is defined by

$$N = \frac{N_{Gr}}{N_{Re}} = \frac{g\,\beta\,A\,a^4}{\nu_o^2} \Big/ \frac{U_o\,a}{\nu_o} \qquad (11)$$

The perturbation expansions about the fully-developed isothermal flow are

$$\phi_1 = \phi_{10}\,(\lambda,\theta) + \sum_{n=1}^{\infty}\phi_{1n}\,(\epsilon,\lambda,\theta)\,N^n \qquad (12)$$

$$\psi = \sum_{n=1}^{\infty}\psi_n\,(\epsilon,\lambda,\theta)\,N^n \qquad (13)$$

$$\rho = \rho_o + \sum_{n=1}^{\infty}\rho_n\,(\epsilon,\lambda,\theta)\,N^n \qquad (14)$$

$$\frac{F}{a\,A} = \Phi = \sum_{n=0}^{\infty}\Phi_n\,N^n \qquad (15)$$

where the stream function is

$$\frac{\rho}{\rho_o}\lambda\,\phi_2 = \frac{\partial\psi}{\partial\theta} \text{ and } \frac{\rho}{\rho_o}\phi_3 = -\frac{\partial\psi}{\partial\lambda} \qquad (16)$$

The boundary conditions are

$$\phi_1\,(1,\theta) = \Phi\,(1,\theta) = 0; \quad \phi_1\,(0,\theta) = \Phi\,(0,\theta) \text{ finite}$$

$$\frac{\partial\psi(1,\theta)}{\partial\lambda} = \frac{\partial\psi(1,\theta)}{\partial\theta} = 0;$$

$$\frac{1}{\lambda}\frac{\partial\psi(0,\theta)}{\partial\theta} = \frac{\partial\psi(0,\theta)}{\partial\lambda} \text{ finite} \qquad (17)$$

and from continuity

$$\int_0^{2\pi}\int_0^1 \frac{\rho}{\rho_o}\phi_1\,\lambda\,d\lambda\,d\theta = \int_0^{2\pi}\int_0^1 \frac{\rho\,u}{\rho_o\,U_o}\lambda\,d\lambda\,d\theta = \pi \qquad (18)$$

Some comment about the density distribution and its consequences seems worthwhile. Combining Equations (6), (7), (14), and (15) one obtains

$$\frac{\rho}{\rho_o} = 1 - (\epsilon + \Phi_o)\frac{N_{Fr}}{N_{Re}}N - \left(\frac{N_{Fr}}{N_{Re}}\right)\sum_{n=1}^{\infty}\Phi_n\,N^{n+1}$$

or

$$\frac{\rho_1}{\rho_o} = -\frac{N_{Fr}}{N_{Re}}(\epsilon + \Phi_o)$$

and it is seen that the Froude number enters the problem as a first-order perturbation. When one collects the coefficients of N and retains only first-order terms, the continuity equation reduces to the system

$$\frac{\partial}{\partial\epsilon}\phi_{10} = 0$$

$$\lambda\frac{\partial}{\partial\epsilon}\left[\frac{\rho_1}{\rho_o}\phi_{10} + \phi_{11}\right] + \frac{\partial}{\partial\lambda}(\lambda\,\phi_{21}) + \frac{\partial\phi_{31}}{\partial\theta} = 0$$

If $\psi_1 = \psi_1\,(\lambda,\theta)$, then it is necessary that

$$\frac{\partial}{\partial\epsilon}\left[\frac{\rho_1}{\rho_o}\phi_{10} + \phi_{11}\right] = 0$$

or

$$\phi_{11} = \frac{N_{Fr}}{N_{Re}}\left[\phi_{10}\,\epsilon + \Phi_o\right] + f(\lambda,\theta)$$

where $f(\lambda, \theta)$ is a function of integration to be determined later. Consequently Equation (18) becomes

$$\int_0^{2\pi}\int_0^1\left[\phi_{10} + fN + \ldots\right]\lambda\,d\lambda\,d\theta = \pi$$

Substituting Equations (12) to (15) into Equations (8) to (10), after equating the coefficients of equal powers of N, one obtains a set of simultaneous equations. Considering only those for n = 0,1 one gets

$$\left.\begin{aligned}\frac{\partial}{\partial\lambda}(\nabla^2\phi_{10}) &= 0\\ \frac{\partial}{\partial\theta}(\nabla^2\phi_{10}) &= 0\end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned}\frac{\partial}{\partial\lambda}\left[\nabla^2\phi_{11} + \frac{N_{Re}}{\lambda}\left(\frac{\partial\psi_1}{\partial\lambda}\frac{\partial}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\cdot\frac{\partial}{\partial\lambda}\right)\phi_{10} - N_{Fr}\,\phi_{10}^2\right] &= \sin\theta\\ \frac{1}{\lambda}\frac{\partial}{\partial\theta}\left[\nabla^2\phi_{11} + \frac{N_{Re}}{\lambda}\left(\frac{\partial\psi_1}{\partial\lambda}\cdot\frac{\partial}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\cdot\frac{\partial}{\partial\lambda}\right)\phi_{10} - N_{Fr}\,\phi_{10}^2\right] &= \cos\theta\end{aligned}\right\} \qquad (20)$$

and

$$\nabla^4\psi_1 = \cos\theta\left(\frac{\partial\Phi_o}{\partial\lambda}\right) \qquad (21)$$

$$\nabla^2\Phi_o - N_{Pe}\,\phi_{10} = 0 \qquad (22)$$

An equation involving ψ_o doesn't appear, since it is identically zero because ψ = 0 when N = 0. The solutions for ψ_1 and Φ_o were given by Morton, and ϕ_{10} is the Hagen-Poiseuille distribution. In the present notation

$$\phi_{10} = 2\,(1 - \lambda^2)$$

$$\Phi_o = \frac{N_{Pe}}{8}(4\lambda^2 - \lambda^4 - 3)$$

$$\psi_1 = \frac{N_{Pe}}{2{,}304}(1 - \lambda^2)^2(10 - \lambda^2)\,\lambda\cos\theta$$

With these results and the expression for ϕ_{11}, Equation (20) becomes

$$\frac{\partial}{\partial\lambda}\left[\nabla^2 f - \frac{N_{Pe}N_{Re}}{576}\lambda(1-\lambda^2)^2(10-\lambda)\sin\theta + \frac{N_{Pe}}{4}\cdot\frac{N_{Fr}}{N_{Re}}(28 - 80\,\lambda^2 + 36\,\lambda^4) - 4N_{Fr}(1-\lambda^2)^2\right] = \sin\theta$$

$$\frac{1}{\lambda}\frac{\partial}{\partial\theta}\left[\nabla^2 f - \frac{N_{Pe}N_{Re}}{576}\lambda(1-\lambda^2)(10-\lambda)\sin\theta + \frac{N_{Pe}}{4}\cdot\frac{N_{Fr}}{N_{Re}}(28 - 80\,\lambda^2 + 36\,\lambda^4) - 4N_{Fr}(1-\lambda^2)^2\right] = \cos\theta \qquad (23)$$

Let

$$f = f_1 + f_2$$

where

$$\left.\begin{aligned}\frac{\partial}{\partial\lambda}\nabla^2 f_1 + 4\frac{N_{Fr}}{N_{Re}}\cdot N_{Pe}\left[9\,\lambda^3 - 10\,\lambda\right] - 16\,N_{Fr}(\lambda^3 - \lambda) &= \sin\theta\\ \frac{1}{\lambda}\frac{\partial}{\partial\theta}\nabla^2 f_1 &= \cos\theta\end{aligned}\right\} \qquad (24)$$

and

$$\nabla^2 f_2 - \frac{N_{Pe}N_{Re}}{576}\lambda\,(1-\lambda^2)^2 (10-\lambda^2)\sin\theta = 0 \quad (25)$$

The solution for an equation similar to Equation (25) was given by Morton and yields

$$f_2 = -\frac{N_{Pe}N_{Re}}{46{,}080}\lambda\,(1-\lambda^2)(49-51\,\lambda^2+19\lambda^4-\lambda^6)\sin\theta \quad (26)$$

Equations (24) may be solved by letting

$$f_1 = P(\lambda) + Q(\lambda)\sin\theta$$

where $P(\lambda)$ and $Q(\lambda)$ satisfy

$$\frac{\partial}{\partial\lambda}\nabla^2 P = -4\,N_{Pe}\frac{N_{Fr}}{N_{Re}}(9\lambda^3-10\lambda) + 16\,N_{Fr}\,(\lambda^3-\lambda)$$

$$\frac{1}{\lambda}\left[\nabla^2 Q - \frac{Q}{\lambda^2}\right] = 1$$

and the result is

$$f_1 = -\frac{1}{8}\lambda\,(1-\lambda^2)\sin\theta + \frac{N_{Fr}}{18}(2\lambda^6-9\lambda^4+9\lambda^2-2) + \frac{N_{Pe}}{24}\cdot\frac{N_{Fr}}{N_{Re}}(6\lambda^6-30\lambda^4+31\lambda^2-7) \quad (27)$$

Combining the preceding results one gets

$$\phi_1 = 2(1-\lambda^2) - \frac{N}{8}\left[1+\frac{N_{Pe}N_{Re}}{5760}(49-51\lambda^2+19\lambda^4-\lambda^6)\right]\lambda(1-\lambda^2)\sin\theta + N\left[\frac{N_{Fr}}{18}(2\lambda^6-9\lambda^4+9\lambda^2-2) + \frac{N_{Pe}}{24}\cdot\frac{N_{Fr}}{N_{Re}}(11\lambda^2-11) + 2\frac{N_{Fr}}{N_{Re}}\epsilon\,(1-\lambda^2)\right] \quad (28)$$

This result is that which one would expect, since the buoyancy effect does not vanish as $N_{Re} \to 0$ where free convection prevails. At higher values of N_{Re}, the second term within the bracket is dominant, whereas for small N_{Re} the first becomes more important. It is interesting, that to the first order, the variation in the axial pressure gradient due to the axial temperature gradient superimposes an additional effect which is independent of the behavior of the r and θ velocity components. For creeping flows one can neglect the inertial terms in the momentum equations, and if it is further assumed that the density is constant except in the body force terms, then the exact solution is

$$\phi_1 = 2\,(1-\lambda^2) - \frac{N}{8}\lambda\,(1-\lambda^2)\sin\theta \quad (29)$$

Equation (29) is identical to the first two terms of the perturbation solution given by Equation (28).

The first-order terms affect the velocity and temperature profiles but not the overall flow or heat transfer properties such as f or N_{Nu} so long as laminar flow is stable. However second-order terms do alter overall heat and momentum transfer.

One may obtain ϕ_{12} as

$$\phi_{12} = \frac{N_{Fr}}{N_{Re}}\{\phi_{11}\,\epsilon + \Phi_o\,\phi_{11} + \Phi_1\,\phi_{10}\} + g\,(\lambda,\theta) \quad (30)$$

When one neglects the small Froude number effect, Equations (8) to (10) and (30) yield

$$\nabla^2 g = N_{Re}\left[\frac{1}{\lambda}\left(\frac{\partial\psi_1}{\partial\theta}\cdot\frac{\partial}{\partial\lambda} - \frac{\partial\psi_1}{\partial\lambda}\cdot\frac{\partial}{\partial\theta}\right)\phi_{11} + \frac{1}{\lambda}\left(\frac{\partial\psi_2}{\partial\theta}\cdot\frac{\partial}{\partial\lambda} - \frac{\partial\psi_2}{\partial\lambda}\,\frac{\partial}{\partial\theta}\right)\phi_{10}\right] + C \quad (31)$$

Equation (31) differs from Morton's Equation (19) in that ψ_2 is affected by axial pressure gradient variations, and the constant of integration C cannot be zero if continuity is satisfied.

The corrected solution to Equation (31) has been obtained in a straightforward manner but is lengthy. However its manifestations can be briefly summarized in terms of the overall resistance to flow reflected by the friction factor f. If f is defined in terms of the overall force on the pipe wall, with $g(\lambda,\theta)$, one obtains

$$f = \frac{8}{N_{Re}}\left[1 + (34.8\,N_{Re}\,N_{Pe} + 0.21\,N_{Re}^2\,N_{Pe}^2)\left(\frac{N}{2304}\right)^2\right] \quad (32)$$

where the new interaction term 34·8 $N_{Re}\,N_{Pe}$ arises and is the same order of magnitude as 0.21 $N_{Re}^2\,N_{Pe}^2$ if roughly $N_{Re}\,N_{Pe} < 2{,}500$. On the other hand the interaction is at least an order of magnitude larger if $N_{Re}\,N_{Pe} < 25$ which may well occur with liquid metals.

To $O(N^2)$ there is a significant interaction between circulation and pressure gradient effects which influences overall and local flow and heat transfer properties. This interaction apparently extends the influence of each effect to lower and higher values of $N_{Re}\,N_{Pe}$ respectively for a given N. Furthermore deviations from Poiseuille flow are not simply related to the single parameter $N_{Ra}\,N_{Re}$ except for small $(t_w - t_b)$ and sufficiently large $N_{Re}\,N_{Pe}$. In the numerical example given by Morton for $N_{Ra}\,N_{Re} = 1{,}500$ the estimate of deviations from the forced flow solution given for water, air, and mercury are too low, and the error increases as N_{Pr} decreases. For mercury under these conditions $N \sim 150$ which seems much too large even for the solutions given here to be valid.

This analysis indicates that terms beyond the second order are necessary particularly for the intermediate range of $N_{Re}\,N_{Pe}$, where the magnitude of N must be small. Although such a calculation is straightforward, it would be very tedious unless an alternate method of solution is devised which employs a computer.

ACKNOWLEDGMENT

This work was sponsored by the National Science Foundation under grant G15016.

NOTATION

- a = tube radius
- A = axial temperature gradient
- g = gravitational component perpendicular to earth's surface
- N_{Ra} = Rayleigh number
- N_{Re} = Reynolds number based on radius
- N_{Pe} = Peclet number based on radius
- N_{Gr} = Grashoff number, $\left(\frac{A\,a^4\,\beta\,g}{\nu_o^2}\right)$
- N_{Fr} = Froude number, $\left(\frac{U_o^2}{a\,g}\right)$
- N = expansion parameter for perturbation solution, $\left(\frac{N_{Gr}}{N_{Re}}\right)$
- t = local fluid temperature
- t_o = average temperature at entrance to heat transfer section
- t_w = wall temperature
- t_b = bulk fluid temperature
- $\vec{V}$ = dimensionless velocity vector referred to U_o
- u,v,w = axial, radial and angular coordinates
- x,r,θ = x, r, and θ velocity components
- U_o = average axial velocity at entrance to heat transfer section
- β = coefficient of expansion
- ϵ,λ = $\frac{x}{a}$, $\frac{r}{a}$
- ϕ_1,ϕ_2,ϕ_3 = $\frac{u}{U_o}$, $\frac{v}{U_o}$, $\frac{w}{U_o}$, dimensionless velocity components

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